Introduction to Econophysics

Look back into the future: How to predict tomorrow's science by the data of yesterday

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Abstract

Predicting the future by yesterday´s data is a problem of all scientific fields. According to the present model the near future contains terms (V) that are valid at all times, and terms (U) that are presently unknown. However, in due time the unknown future will flow into the known past, and we may identify the terms (V) and (U). In economics, (V) and (U) correspond to *ex ante* and *ex post*, in physics to *conservative* and *not conservative*. Surprisingly, we find the same (V) and (U) structure in many mathematical fields: in calculus, we have *exact* and *not exact* differential forms or *Riemann* and Stokes integrals, in statistics *real* and *probable* terms, in differential equations *linear* and *non-linear* equations and in the space of real numbers *rational* and *irrational* numbers. Apparently, we may represent the (V) and (U) structure in social and natural sciences by common mathematical instruments. We may investigate the new interdisciplinary field *econophysics* by calculus and probability theory, by non-linearity and complexity, by system dynamics and by chaos theory.

Introduction

Econophysics is a new field that investigates economic problems with the methods of physics. Eugene Stanley has coined the name in 1993. The book "Introduction to Econophysics" by Mantegna and Stanley (2000) discusses mainly correlations and complexity in finance; this is until today the main application of econophysics. However, in 2001 the German Physical Society founded the group "Physics of socio-economic Systems". This group aims at a far wider scope, at the application of physics to socio-economic problems. This includes macroeconomics, microeconomics, finance, socio-economics, complexity, nonlinear systems, chaos and other topics. In the following time, similar groups have formed worldwide under various names, and a number of books have been published in this new field. [1 – 5]. The present paper discusses a future model that explains the different approaches to econophysics like calculus, statistics, nonlinearity and chaos theory. The main focusses is, however, the application of calculus to macroeconomics.

Mainstream economists for several reasons do not yet accept the new field econophysics. The language and thinking of economists and physicists is quite different, economics is a more philosophical experience, whereas physics relies on mathematical theories. There are many misunderstandings on both sides and so far, economists claim, econophysics has not generated any striking new results. However, physicists point out that the agreement of experience and theory, of social and natural science is already a big success! However, econophysics also shows many errors and weaknesses of mainstream economics. In addition, banks have started to hire physicists for bank services like portfolio management.

1. The U – V future model

Why do we send our children to school and teach them the knowledge of yesterday, of our past. Will they be able to make any use of their learning in the future? Well, in learning the past, we hope that some of our knowledge today will still be valid in the near future. Of course, children will encounter many new and unknown things in their future lives. How will these unknown facts interact with the experience of yesterday? This is the starting point of the present U – V future model.

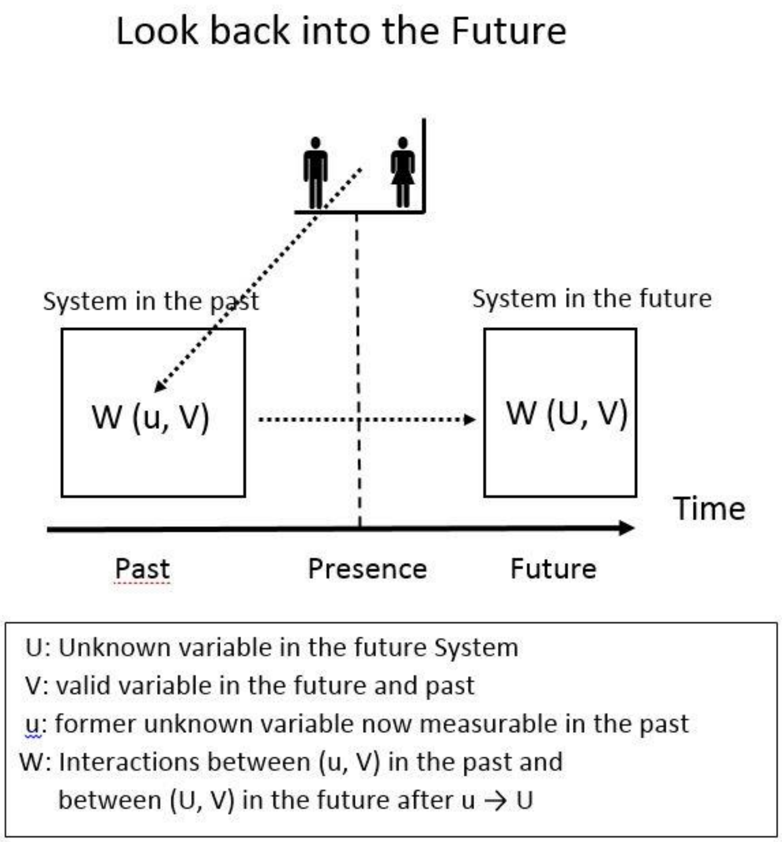

Figure 1. Researchers cannot look into the future. Scientists use the past like a mirror

Future contains elements that are always valid (V) and elements (U) that are unknown. These elements interact in the near future by an unknown relationship W (U, V). However, is it possible to find out this relationship today? The answer is "yes", all we have to do is to wait. After due time the near future will flow into the past. The elements (V) remain the same. The elements (U) become measurable in the past and we may now call them (u). If the future is not too far away, the relationship will still be the same and is now W (u, V), figure 1. Researchers have to distinguish between the measurable elements (u) that will be unknown in the future, and the elements (V) that are always valid, and then they have to determine their relationship, W (u, V). The scientists have to present the relationship W (u, V) in a proper mathematical apparatus that will preserves and transfers the (U, V) structure into the future. This will be the most difficult part, but we will see that many sciences including mathematics already have a U – V structure.

Six examples for the U – V future model

We now look for examples of (U) – (V) structures in science and mathematics

1. Physics with conservative (V) and not conservative (U) forces

In physical sciences, we have two different kinds of systems, *conservative* or *non-conservative*.

V: conservative systems are free from friction; the future is predictable. The laws of physics without friction like in mechanics are W (0, V).

When Kepler observed the movement of planets in space, he was lucky that space does not contain many other particles that may create non-conservative frictional forces: U = 0. Accordingly, the Kepler laws W (0, V) for planets are valid in the past and in the future.
Newton found the laws of mechanics W (0, V); which are valid, if friction is negligible or small.

U: Non- conservative frictional systems make the future unpredictable. Friction creates heat and leads to the laws of thermodynamics, W (U, V).

2. Economics with ex ante (V) and ex post (U) terms

Investors know that it is not possible to predict the outcome of their investments like the flight of an arrow, which we may calculate before it hits the goal. In economics, we may divide all terms into two categories: *ex ante* and *ex post*.

V: Ex ante terms like a function F(x, y) are valid (V) in the past and future, we may calculate functions at any time. Other examples are contracts. If we have an annual contract for our savings account, we may calculate the interest in advance.

U: Ex post terms like income are only known after we have earned the money. We can file our income tax only at the end of the year, not in the beginning. Even if we have an annual work contract, we do not know what income or loss may come up. Another example of an *ex post* term is the value of our shares at the stock market.

3. Two-dimensional calculus with exact (V) and not-exact (U) differential forms

In addition to social and natural sciences, many mathematical fields show a (U) – (V) structure. In calculus with two variables, we have two kinds of differential forms:

(V): Exact differential forms (d F) have a stem function (F), which we may calculate any time.

(U): Not exact differential forms (δ M) do not have a stem function (M), and we may not calculate them, as long as the information is incomplete. We will discuss exact and not-exact differential forms and their application to economics in more detail in the following chapters.

4. Line integrals by Riemann (V) and Stokes (U)

In two-dimensional calculus, we have two different types of line integrals by Riemann and Stokes.

(V): Riemann integral: The line integral of an exact differential form (d F) is a Riemann integral. The integral does not depend on the path of integration and we may calculate the Riemann integral for any boundaries A and B. The closed Riemann integral is always zero, the integrals from A to B and from B to A cancel, as the Riemann integral is path independent.

(U): Stokes integral: The line integral of a *not-exact* differential form (δ M) is a Stokes integral, which depends on the path of integration. The closed Stokes line integral is not zero, the path dependent integrals from A to B and back from B to A do not cancel. We will also discuss Riemann and Stokes integrals and their application to economics in more detail in the following chapters.

5. Stochastic theory with "real" (V) and "probable" (U) values

In stochastic theory, we have two kinds of terms: *real* and *probable* values.

(V): Real functions are often the constraints in a stochastic theory, and they are generally well known.

(U): Probable terms depend on the system and we may not calculate these terms without further knowledge.

6. Linear (V) and non-linear (U) differential equations

Social scientists sometimes apply differential equations to explain social models. There are two kinds of differential equations:

(V): Linear differential equations may be solved by various methods.

(U): Non-linear differential equations often cannot be solved in general.

Four mathematical tools that may represent economics

Classical economics is a philosophy based on the ideas of Adam Smith and his successors. Many economists wonder whether macroeconomics with so many unknown or even chaotic parameters will ever be an exact science with a solid mathematical base.
However, the future model tells us, that many sciences and mathematical fields have a similar U – V structure of known (V) and unknown (V) elements. Apparently, we have several possibilities to write down economic theory in mathematical terms; by differential forms, line integrals, stochastic theory, differential equations and chaos theory.

1. Differential forms in macroeconomics

We may chose differential forms in two dimensions as a mathematical tool in economic theory: all ex ante terms correspond to exact differential forms, and all ex post terms to not-exact differential forms.

2. Line integrals in macroeconomics

Alternatively, we may chose two-dimensional line integrals as a mathematical tool in economic theory: all ex ante terms are Riemann integrals, and all ex post terms are Stokes integrals.

Neoclassical theory employs calculus in one dimension, which contains exact differentials and Riemann integrals. Not-exact differential forms and Stokes integrals are not used and mostly unknown in mainstream economics. The basis of neoclassical macroeconomics is the Solow model: output and income are functions of capital and labor, Y = A F (K, L). The factor (A) reflects the advancement of technology. This law is widely used in neoclassical theory. However, the Solow model violates the U – V structure of the future model: Income is an ex post or (U) term, and should not be a function, but a not exact differential form or a Stokes integral! Functions (F) are *ex ante* or (V) terms and should be exact differential forms or

Riemann integrals! The Solow law: Y = F cannot be valid, ex post cannot be ex ante, U cannot be equal V! For this reason, we will discuss calculus-based economics in more detail in this paper.

3. Stochastic theory in microeconomics

If we chose stochastic theory as a base of economics, all ex ante terms must correspond to real functions, and all ex post terms to probability terms. Microeconomics and finance are using stochastic theory [6], and will not be discussed in this paper.

4. Non-linear differential equations and chaos theory

Economic theory in differential equations requires ex ante and ex post terms to correspond to linear and non-linear differential equations, which are generally not solvable. They are the basis of system science, complexity and chaos theory [7, 8], and will not be discussed here, any further.

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2. Calculus in Economics

Economics depends on two production factors, capital and labour. Accordingly, this is a short introduction to calculus in two dimensions by differential forms. In two dimensions, we have two kinds of linear differential forms – exact and not exact – and two kinds of line integrals, the Riemann and the Stokes line integral. The mathematical terms "exact" and "not exact" correspond to "ex ante" and "ex post" in economics. The examples in this chapter are one key to a basic understanding of the present approach to economics. For further reading, one may refer to textbooks on calculus or on differential forms, [9, 10].

The two linear differential forms

Exact differential forms (ex ante)

We may look at an arbitrary function, F (x, y) – perhaps a production function – and build the derivatives:

Exact differential forms	*Example 2.1*
$F = F(x, y)$	$F = x^4 y^7$
$dF = F_x\, dx + F_y\, dy$	$dF = 4x^3 y^7\, dx + 7x^4 y^6\, dy$
$\mathbf{F_{xy} = F_{yx}}$	$28x^3 y^6 = 28x^3 y^6$

The mixed second derivatives of an *exact* differential form will *always be equal*. This is the important property of all exact differentials.

Not exact differential forms (ex post)

A not exact differential form is marked by "δ" like (δ M), and may be constructed by multiplying an exact differential d F (x, y) by a factor λ (x, y). If we take as an example the exact differential form d F (x, y) in *example 2.1* and multiply d F (x, y) by the factor (x), we obtain a not exact differential form (δ M):

Not exact differential forms	*Example 2.2*
$\delta M = x\, dF(x, y)$	$\delta M = x(4x^3 y^7\, dx + 7x^4 y^6\, dy)$
$\delta M = M_x\, dx + M_y\, dy$	$\delta M = 4x^4 y^7\, dx + 7x^5 y^6\, dy$
$M_{xy} \neq M_{yx}$	$28x^4 y^6 \neq 35x^4 y^6$

The mixed second derivatives of *not exact* differential forms will *not be equal*, as they are incomplete. In economic theory, inexact differentials are called *ex post* terms.

The integrating factor

In the previous chapter, we have constructed the not exact differential (δ M) by multiplying the exact differential form (d F) in example 2.1 by the factor x:

$$\delta M = x\, dF(x, y). \qquad (2.1)$$

We may reverse this procedure: A not exact differential form (δ M) may be turned into an exact differential form (d F) by an integrating factor 1/ x,

$$d F = \delta M / x \qquad (2.2)$$

The integrating factor in this example is (1 / x). In general, the integrating factor of a not exact differential form (δ M) will be a function λ (x, y),

$$\delta M = \lambda (x, y)\, d F (x, y) \qquad (2.3)$$

We may replace a not exact differential form (δ M) by an exact differential form (d F), multiplied by an integration factor λ(x, y). This integrating factor always exists in two-dimensional calculus.

The two line integrals

Closed Riemann line integrals in the x – y plane

We may integrate Riemann integrals along a closed line; these integrals do not have borders. The closed line of integration may be a circle, a rectangle or any other closed line without intersecting. A closed Riemann integral may be split into two parts, the first part is the integration from A = (0, 0) to B = (x, y) on one path, and the second part the integration from B to A along another path:

$$\oint d\,F\,(x, y) = \int_A^B d\,F + \int_B^A d\,F \qquad (2.4)$$

Reversing the integral limits of the second integral will change the sign of the integral.

$$\oint d\,F\,(x, y) = \int_A^B d\,F - \int_A^B d\,F \;= 0 \qquad (2.5)$$

The closed Riemann line integral is always zero. The Riemann integral is independent of the path of integration.

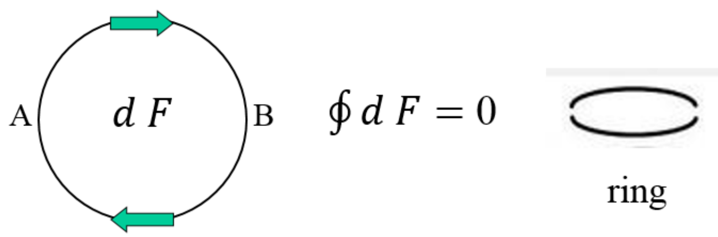


Fig. 2.1. A closed Riemann line integral follows the path of a ring. After each round the integral arrives again at the starting point, the closed integral is zero!

In fig. 2.1, the closed Riemann integral follows a cyclic path, a ring, after each round the integral arrives again at the starting point, the closed integral is zero! We arrive at the same result, if the path of integration is rectangular. The closed Riemann integral is zero; this is independent of the shape of the path of integration.

Closed Stokes integrals in the x – y plane

We may integrate Stokes integrals along a closed line; these integrals do not have borders. The closed line of integration may be a circle, a rectangle or any other closed line without intersecting. A closed Stokes line integral may be split into two parts, the first part is the integration starting from A = (0, 0) to B = (x, y), and the second part is the integration from B to A along another path:

$$\oint \delta\,\mathrm{M}\,(x,y) = \int_A^B \delta\,\mathrm{M} + \int_B^A \delta\,\mathrm{M} = \int_A^B \delta\,\mathrm{M} - \int_A^B \delta\,\mathrm{M} \;\neq 0 \tag{2.6}$$

Reversing the integral limits of the second integral will change the sign of the integral.
But the Stokes integrals depend on the path of integration. Accordingly, closed Stokes line integrals are not zero, they are either larger or smaller than zero,

$$\oint \delta\,\mathrm{M}\,(x,y) = \int_A^B \delta\,\mathrm{M} - \int_A^B \delta\,\mathrm{M} = \Delta\,\mathrm{M} > 0 \quad \text{or} \quad \Delta\,\mathrm{M} < 0 \tag{2.7}$$

We may interpret the closed Stokes line integral as a curl or spring, fig. 2.2.

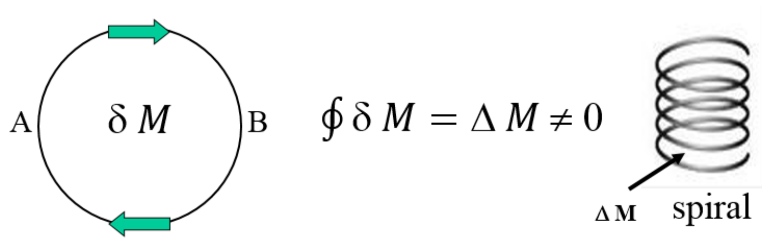
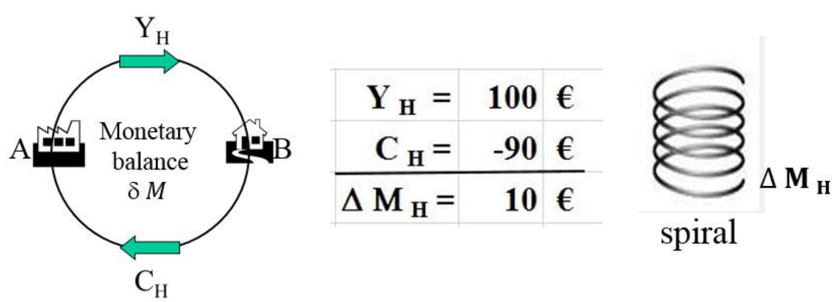


Fig. 2.2. A positive closed Stokes integral is a curl or a spiral that winds up, and a negative Stokes integral is a spiral that winds down. Δ M is the lift after each round.

In fig. 2.2, the closed Stokes integral follows a curl or spiral, after each round the spiral winds up or down by the amount of Δ M. The path of integration may be a circle, fig. 2.2 or rectangular.

Example: A closed Stokes line integral as an unbalanced account.

A household (H) works in industry (In) earning 100 € per day and spending 90 € for food and goods. The surplus is 10 € per day.



Stokes: $\oint \delta\, M \quad = \quad Y_H - C_H \quad = \quad \Delta\, M_H$

Fig. 2.3. A monetary account of a household working in industry is a closed Stokes integral

In economics, an unbalanced account, fig. 2.3, is a good example for a closed Stokes line integral. A household that works in industry earns €100 per day and spends €90 for food and goods. The area within the monetary balance is the surplus is €10 per day. The surplus indicates that we may regard the monetary circuit as closed Stokes line integral. In the second picture on the right side of fig. 2.3, the surplus corresponds to the spiral. The monetary account of the household spirals up by €10 every day. Apparently, we may regard a monetary balance, economic circuits and unbalanced accounts as closed Stokes line integrals.

3. Calculus based Economics

In the following chapters, we will discuss the application of differential forms and integrals to economic science and accounting.

Double entry accounting

Luca Pacioli (1445 – 1517) introduced double entry bookkeeping, the basis of modern accounting. Table 1 shows the monetary and the productive accounts of a household of fig. 2.6, working in industry.

A household (H) works in industry (In) earning 100 € per day and spending 90 € for food and goods. The surplus is 10 € per day.

(monetary units: €) *(originally energy units, kcal, MJ)*

Y_H =	100	€	W_H =	-100	€
C_H =	-90	€	G_H =	90	€
ΔM_H =	10	€	ΔP_H =	-10	€

monetary account + productive account = 0

Fig. 3.1: Double entry book keeping requires always two accounts that will add to zero. Fig. 3 1 shows the double entry account (δ M) of a household working in industry, earning 100 € and consuming 90 € per day. The monetary account contains income and consumption costs, and grows by 10 € per day. The productive account (δ L) contains labour input and consumption goods and loses 10 € per day, the sum of both accounts is always zero.

Double entry accounting requires always two accounts that will add to zero. The monetary account (δ M) contains income and consumption costs, and grows by 10 € per day. The productive account (δ L) contains labour input and consumption goods and loses 10 € per day, the sum of both accounts is always zero.

Natural science measures the productive account in energy units. We measure labor or work in mega joule and food as the main consumption good in kilocalories. In contrast, double entry accounting measures the productive account in monetary units. We will come back to this contrast in natural and social science at a later point.

Double entry accounting in Stokes integrals

Unbalanced accounts are closed Stokes line integrals and accordingly, we may write double entry accounting as Stokes integrals,

$$\oint \delta M + \oint \delta L = 0 \qquad (3.1)$$

Monetary account + productive account = 0

The Stokes integrals in eq. (3.1) are equivalent to the monetary and productive accounts in table 1. In the next paragraph, we will find most important applications of closed Stokes line integrals in double entry accounting.

The law of macroeconomics in integral form

When Luca Pacioli introduced double entry bookkeeping in 1495, he established the basis of modern accounting, which is still valid after more than 500 years. Accounting is the basis of all economics, and we may call the Stokes equation (3.1) of double entry accounting the fundamental principle of economics,

$$\oint \delta M = - \oint \delta L \qquad (3.2)$$

The monetary circuit (δ M) measures the production circuit (- δ L) in monetary units. This is the fundamental law of economics: Profit (δ M) comes from labour input (- δ L)! This fundamental law agrees with the conditions of the U – V future model: all ex post terms like surplus (δ M) and labour input (δ L) are Stokes integrals. The future model has succeeded to lead us to economic laws that will be valid in the future! We will discuss the fundamental law (3.2) in more detail in the following chapters [11, 12].

The laws of macroeconomics in differential forms

We will now derive the laws of economics in differential forms by applying calculus of chapter 2.

The first law of economics

The first law of economics in differential forms reads

$$\delta M = d K - \delta L \qquad (3.3).$$

If we integrate eq. (3.3) by closed Stokes integrals, this leads back to eq. (3.2). The new term (d K) is an exact differential form and the closed Riemann integral of (d K) will be zero. Eq. (3.3) contains three differential forms with the common dimension *money*:

1. The not exact monetary term (δ M) stands for money, for surplus or losses and relates to a person, a household, a company, a country or any economic system.
2. (δ L) is the *ex post* term of labour input or production of persons, households, companies, countries or any economic system.
3. (d K) is an exact term with the dimension “money”. (K) refers to money or capital.

All economic terms in eq. (3.3) are measured in €, US $, £, ¥ or other monetary units.

The second law of economics

According to calculus in chapter 2 we may transform a not exact differential form (δ M) into an exact differential form (d F) by an integrating factor λ,

$$\delta M = \lambda \, d F \qquad (3.4).$$

This is the second law of economics. (δ M) is the inexact or *ex post* surplus or loss. The function (F) is the production function and eq. (3.4) is the proof of existence of production functions in economics. The dimension of the integrating factor (λ) depends on the production function (F). We will find the production function to be a number, and then the integrating factor (λ) has a monetary dimension. We will discuss the second law in more detail, below. However, we may now come back to the critics of neoclassical theory in the first chapter:

Neoclassical theory employs calculus in one dimension, which contains exact differentials and Riemann integrals. Not-exact differential forms and Stokes integrals are not used and mostly unknown in mainstream economics. The basis of neoclassical macroeconomics is the Solow model [13]: output and income are functions of capital and labor, Y = A F (K, L). The factor (A) reflects the advancement of technology. This law is widely used in neoclassical theory. However, the Solow model violates the U – V structure of the future model: Income is an ex post or (U) term, functions (F) are *ex ante* or (V) terms! The Solow model: Y = F cannot be valid, ex post cannot be ex ante, U cannot be equal V!

The second law replaces the erratic Solow equation Y = A F (K, N). According to fig. 2.3 (δ M) is the difference between income and costs, and we may replace positive values of (δ M) by income (δ Y). The second law now becomes

$$\delta Y = \lambda \, d F \qquad (3.4\ a)$$

Eq. (3.4 a) solves the Solow dilemma of ex ante and ex post terms: the ex post term income is represented by a not-exact differential form (δ Y), and the function F as an ex ante term is represented by the correct exact differential form (d F).

The third law of economics

The second law replaces the inexact differential form (δ M) by an exact differential form (d F) and an integrating factor (λ). In the same way we may replace the inexact differential form (δ L) of production or labour input by the exact differential (d V) and the integrating factor (P),

$$-\delta L = P \, d V \qquad (3.5).$$

Eq. (3.5) is again a formal result of calculus. (δ L) is the labour input measured in monetary units.. The minus sign indicates that labour must be invested. Yakovenko [14] has connected labour and production (δ L) to goods, to the price (P) and the volume or amount (V) of produced goods.

The price per item (P) has a monetary dimension of money per item, this is measured again in €, US $, £, ¥ etc. We may measure the amount (V) in various units: we can buy commodities by the piece, by kg, by litres or by any other measure that will define an amount.

Again, we have succeeded to find the laws of economics in agreement with the U – V future model. In eqs. (3.3) to (3.5) all economic terms are differential forms, ex ante terms are exact and ex post terms are not-exact differentials.

The integral law of economics in energy units

Before going into a deeper discussion of the new economic laws (3.3) to (3.5), we will look again at double entry accounting. The fundamental principle of double entry accounting states:

The monetary circuit (δ M) measures the production circuit (- δ L) in monetary units.

However, since we measure production in energy units, it would also be possible to turn the statement around:

The productive circuit (-δ L) measures the monetary circuit (δM) in energy units!

Indeed, we can turn monetary units into energy units e.g. by the international oil price (2016):

1 US $ = 118 MJ = 33 kWh

Now we can measure the monetary circuit (δ M) and the production circuit (- δ L) in energy units. But in order to indicate the new units of energy, we replace the letters for surplus (δ M) and production (δ L) of eq. (3.2) by the energy terms of surplus (δ Q) and work input (-δ W),

$$\oint \delta Q = -\oint \delta W \qquad (3.6)$$

The differential laws of economics in energy units

In the same way, we may now transform the laws (3.3) to (3.5) into energy units,

$$\delta Q = d E - \delta W \quad (3.7).$$

$$\delta Q = T d S \quad (3.8).$$

$$\delta W = - P d V \quad (3.9).$$

The economic laws in eqs. (3.7 to 3.9) are identical to the laws in eqs. (3.3 to 3.5), measured in energy units. On the other hand, they are the laws of thermodynamics. Apparently, economics and thermodynamics have the same mathematical structure! Table 1 shows the corresponding terms in economics and thermodynamics.

Symbol	Economics	Unit		Symbol	Thermodynamics	Unit
M	Profit, loss	€, $, £. ¥	↔	**Q**	Heat	kcal, kWh
K	Capital	€, $, £. ¥	↔	**E**	Energy	kcal, kWh
P	Labour, production	€, $, £. ¥	↔	**W**	Work	kcal, kWh
λ	Mean capital	€, $, £. ¥	↔	**T**	Temperature	kcal, kWh
F	Production function	-	↔	**S**	Entropy	-
N	Number	-	↔	**N**	Number	-
p	Price per item	€, $, £. ¥	↔	**p**	Pressure	kcal / m^3
V	Volume, amount	-	↔	**V**	Volume	m^3

Table I Corresponding terms in economics and thermodynamics. We will discuss the different terms in the following chapters.

Profit corresponds to heat; capital to energy, labour to work. The Cobb Douglas production function {15] is replaced by the entropy (S) of the economic system. The integrating factor corresponds to temperature and we may regard (λ) as the "economic temperature" or the mean capital of the economic system. We will discuss the other terms in the following chapters.

The corresponding structures of thermodynamics and economics will make it possible to compare results of economics to those of physics, chemistry or metallurgy. Moreover, it will be easier to look for new economic solutions, when there are solutions already available in other fields.

We will now investigate, whether there are there any indications for this correspondence in economic data.

Fig. 3.2 shows the GDP per capita and energy consumption per capita for the 126 largest countries in the world.

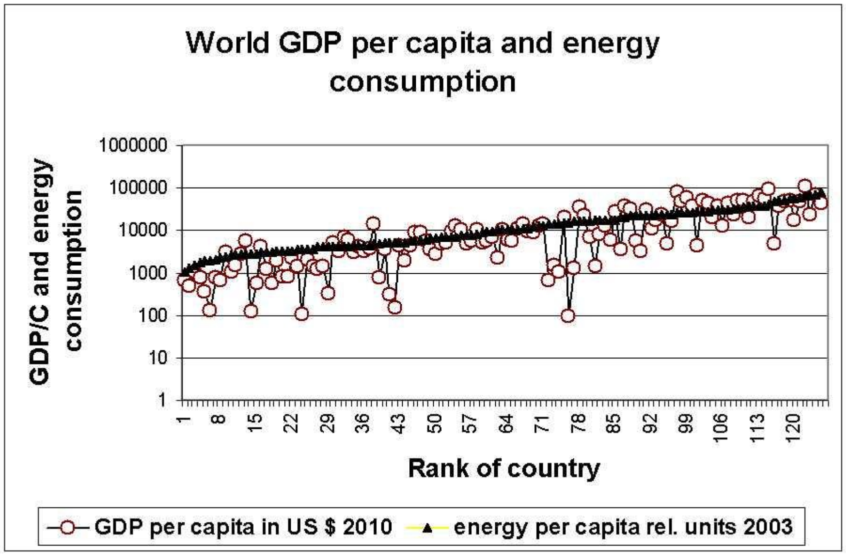


Fig. 3.2 GDP per capita and energy consumption per capita follow the same line for most of the 126 largest countries in the world. Mean capital (K / N) and mean energy consumption (E / N) are equivalent and proportional to the standard of living (λ) of a country, [16].

GDP per capita and energy consumption per capita follow the same line for nearly all large countries in the world. Mean capital (K / N) and mean energy consumption (E / N) are equivalent and proportional to the standard of living (λ) of a country:

$$K / N = c \lambda \tag{3.10}$$

The mean energy is proportional to temperature (T), the integrating factor of the second law of thermodynamics. The mean capital, the GDP per capita, is proportional to the standard of living, (λ), the integrating factor of the second law of economics. The standard of living (λ) is the economic temperature of a country. The factor c is the specific heat factor in thermodynamics and reflects the different ways of storing energy. We may be call c specific capital factor in calculus-based economics and we expect c to reflect the different ways of storing capital.
Fig. 3.2 shows additional information: the world is not at equilibrium, the economic temperature varies by a factor of 1000.

4. Entropy as the new production function

One of the most important results of chapter 7 is the equivalence of production function (F) and entropy (S). Nicolas Georgescu-Roegen (1906 – 1994), a Romanian mathematician and economist, was one of the first scientists to point out the close relationship between economics and physics in his book “*The Entropy Law and the Economic Process”*, (1971) [17].

Entropy and probability

According to Boltzmann entropy (S) is closely connected to probability (p),

$$S = \ln (p) + S_0 \qquad (4.1).$$

S_0 is a probability constant. The results of Boltzmann, Shannon and others will be discussed in chapter XX. We will now only discuss a few important aspects.
The future of any economic system, the next step is the more probable state. Probability will always grow, until it reaches a maximum. Entropy as logarithmic probability will do the same,

$$S \rightarrow \text{maximum} \qquad (4.2).$$

A simple example may be a perfume bottle. After opening the bottle, it is very improbable for perfume to stay within the small volume of the bottle. It is much more probable for the perfume to flow into the open space with a wide distribution, fig. 4.1.

Fig. 4.1. If we open a perfume bottle, the perfume will leave the bottle to occupy the environment and it will never return into the bottle.

Many calculations of modern economics are based on entropy maximization [14]. Entropy does not depend on the kind of elements, only on the number of elements. For this reason, entropy may be applied to any field with a number of objects, to atoms in physics, to goods in economics, to people in social sciences.

Entropy and disorder

From statistics, we may derive another property: entropy as a measure of disorder. We may be explain this by looking at the toys in the children's room. Each toy will have its well-defined place in the room. However, after the children start playing there will be a very small chance that every toy is in its place, again. Generally, the playrooms will be in disorder, in chaos, after the children leave. We will come back to this example in the chapter on probability theory.
We may look at another example of disorder, an empty paper basket in a park, fig. 4.2.

Fig. 4.2. Empty paper basket in a park: the wind has distributed paper and leaves.

1. *Thermodynamics:* The first and second law of thermodynamics tells us, what happened. Combining eqs. (3.3) and (3.4) we get:

$$T\, d S = d E - \delta W \qquad (4.3).$$

A light breeze with the energy (d E) in a park will easily empty a paper basket and generate more and more disorder (d S). The wind distributes the paper throughout the park and will never bring it back into the basket.

However, a janitor may work (δ W) and sweep the paper together and put it back into the basket. Work reduces disorder: (+ d S) means creating disorder or distributing items, whereas (- d S) means collecting, ordering.

2. *Economics:* accordingly, we may combine the first and second law of economics,

$$\lambda\, d F = d K - \delta L \qquad (4.4).$$

A slight hunger at noon may easily empty your purse (d K) and distribute (+d F) your money in the restaurants. The money will never come back into the purse. However, in the afternoon you may go back to work (δ L) in the office and the money will come back into the purse on payday.

Entropy and trade

A positive entropy change (+ d S) means distributing like paper or leaves in a park, paying money to employees, donating money to the poor, or sending goods from a production plant to local shops. In trade (+ d S) means distributing money or buying, distributing goods or selling.

In the same way a negative entropy change (- d S) means collecting money or earning, collecting goods or buying. Entropy is closely related to trade and does not depend on the type of elements, but only on the numbers.

Entropy and production

Eq. (4.4) may be solved for production (δL):

$$\delta L = d K - \lambda d F \quad (4.5).$$

Labor, work, production (d L) is linked to two functions, capital (d K) and the new production function entropy (d F). Labor has two effects:

1. Labor (d L) creates capital (d K). This is why most people go to work, they need money to buy food, clothes and supplies. Labor also creates goods (d K) with a certain capital value.

2. Labor reduces disorder ($-\lambda$ d F), work means ordering: In an automobile plant workers and robots put auto parts together in the right order, a secretary types letters in the correct order, a policeman orders traffic and looks after law and order.

Fig. 4.3 Production of automobiles requires the ordering of many parts: screws, nuts and bolts, wires, tires, wheels etc. All parts have to be placed in the correct position and in the right order, the entropy is reduced, d F < 0.

Entropy reduction also applies to mental work: Mental work orders ideas or puzzles:

Brain work: g+i+r+r+n+o+d+e $\rightarrow$ ordering

Medical doctors order deficiencies within a body, teachers order or develop the minds of young people. Homemakers have known for long times that making order is hard (-unpaid) work! This may be one reason that today most women prefer to work as professionals outside of the house.

5. Trade

In chapter 3 we have derived the differential equations of economics,

$$\oint \delta M = - \oint \delta L \quad (3.2)$$

$$\delta M = d K - \delta L \quad (3.3).$$

$$\delta M = \lambda d F \quad (3.4).$$

$$\delta L = - P d V \quad (3.5).$$

We will now apply these laws to domestic and foreign trade. A trade company has a certain capital (K) to buy and sell a certain amount (V) of commodities, e.g. apples. The merchant buys the products at a lower price (P_1) per item and sells them at a higher price (P_2) per item. The economic equations (3.2) and (3.5) lead to the closed trade circuit:

$$\oint \delta M = - \oint \delta L = + \oint P \; \delta V = (P_2 - P_1) \, \Delta V = \Delta M \quad (5.1)$$

The profit of trade depends on product value (P d V), not on the capital (d K), the closed integral of the exact differential (d K) is zero. Profit (Δ M) is determined by the difference in prices for buying and selling times the amount (Δ V) of products that have been sold.

Trade is a two level process: buying cheap at (P_1) and selling more expensive at (P_2). Fig. 5.1 shows the closed circuit (Stokes integral) of trade.

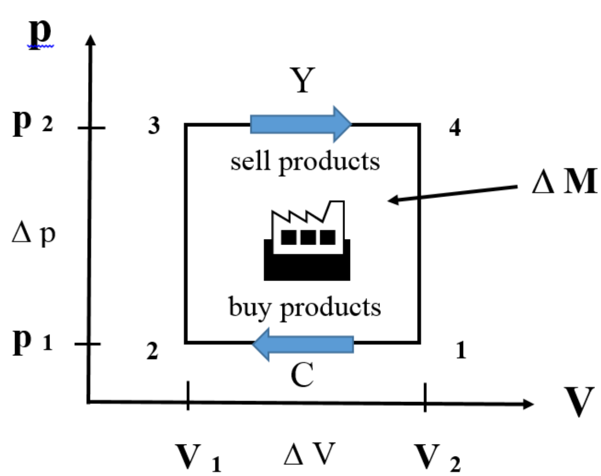


Fig. 5.1. Trade is a two level process: buying cheap at (P_1) and selling more expensive at (P_2). The profit (ΔM) of a trade company is based on inequality of prices. In order to make a profit the trade company has to keep up the difference in prices by separating wholesale and retail prices. Trade companies accomplish the price difference by separating producers and customers by a long trade distance or by market laws. The industry logo indicates the owner of the profit (inside area) of the closed cycle.

We will now discuss equation 5.1 and fig. 5.1 in more detail:
1 → 2: At the farm (f): A merchant enters a farm with the capital (money) $K^f{}_1$ in his pocket. He buys an amount (ΔV) of apples at a cheap price (P_f) per volume and pays $\Delta K^f = P_1 \Delta V$.
*Capita*l: The merchant has now less capital (money) in his pocket, he has traded part of his money ΔK^f for the value of apples at the farm. The merchant´s costs are $C_M = P_1 \Delta V = \Delta K^f$.
Wealth: By trading at the farm, the wealth of the merchant (wealth = capital + value of goods) has not changed. Trade conserves the wealth of buyers and sellers; nobody is robbed in a fair trade!

2 → 3: The merchant brings the products from the farm a market. Here the price of apples (P_2) is higher than at the farm. By changing the location, the merchant has raised the value of the apples. Earning money by trade requires two price levels: cheap – expensive, wholesale – retail, farm – market. Without these two levels, a profit is not possible! In former times, merchants sent ships to far distant places to buy precious goods like spices at a cheap price to sell it back home for a high price. Today merchants buy in China and sell in USA.

3 → 4: At the market (m): the merchant sells his amount (ΔV) of apples to his customers at a higher price (P_2) per volume. The value of his apples at the market is $\Delta K^m = p_2 \Delta V = Y_M$ is the income of the merchant.
Wealth: Again, the wealth of merchant and customer stays constant.
Capital: The merchant has now more capital (money) in his pocket; he has traded his apples for money ΔK^m, the value of apples at the market. The merchant´s income is $Y_M = P_2 \Delta V = \Delta K^m$.

The profit of the merchant is the difference between income and costs,

$$\Delta M = Y_M - C_M = (P_2 - P_1)\, \Delta V, \qquad (5.2)$$

as calculated in eq. (5.2). In the beginning of each trade circle, the merchant has to invest some of his capital to buy the products. But at the end he will get his capital back with a surplus (ΔM). Accordingly, the capital remains untouched after each trade circle; the closed integral of ($d K$) is always zero. Profit is generated only by work δL, not by capital ($d K$)! Trade is work, bringing goods from one (cheap) place to another (more expensive) place.

A company could as well do business without any capital (K) by borrowing money from the bank. At the end of the business circuit, the company repays the money (K) with some interest, which will raise the costs. This reduces the profit of the company and the company has to make sure that its trade model can still survive, $Y - C = \Delta M > 0$!

In neoclassical theory, there are two contradicting models of trade:
1. *The American school claims*, the value (wealth) is conserved in trade. Nobody would go to the market, if he feels being robbed.
2. *The Austrian school claims*, each trading partner values what he gets higher than what he gives away. Otherwise, he would not engage in the trade.
The answer from econophysics: Both schools are right: Wealth (value) is conserved in trade. However, a trader is not interested in wealth (values), but in profit. The merchant is interested in cheap apples at the farm, and in money at the market to make his profit.

6 Production

The production circuit δ L (λ, F)

The productive cycle of a company has four parts: Cheap production, export, expensive sales, recycling. The productive cycle is linked to the monetary cycle: Cheap production leads to low costs. Expensive sales lead to high income. You can look either at the monetary cycle or at the productive cycle.

The Carnot work process of a motor has four parts: Slow (isothermal) compression at low temperature, fast compression, slow expansion at high temperature, fast expansion.
The work cycle is linked to the heat cycle: Slow compression at low temperature leads to low heat generation, slow expansion at high temperature leads to higher heat losses. You can look either at the work cycle or at the heat cycle.

We calculate production by the closed integral

$$-\oint \delta L = \oint \delta M = \oint \lambda \, d\, F \qquad (6.1).$$

Eq. (6.1) indicates that we may use the production factors λ and F; and we derive the differential form δ L (λ, F) from eqs. (3.3) and (3.10),

$$\delta L\,(\lambda, F) = d\,K - \delta\,M = d\,K - \lambda\, d\,F = c\,N\,d\,\lambda - \lambda\,d\,F \qquad (6.2)$$

Productive and monetary circuits are now depending on the production factors (λ) and (F): the factor (λ) is proportional to the value or price and (F) is the entropy: (+d F) corresponds to distributing or selling products and (- d F) to collecting or buying the products. The integrals are not exact and depend on the path of integration. Carnot proposed an ideal path of integration, which was first used to explain the steam engine: integration along (F) at constant (λ) and then integration along (λ) at constant (F). In this way, we obtain four different lines of the line integrals eq. (6.1) in fig. 6.1.

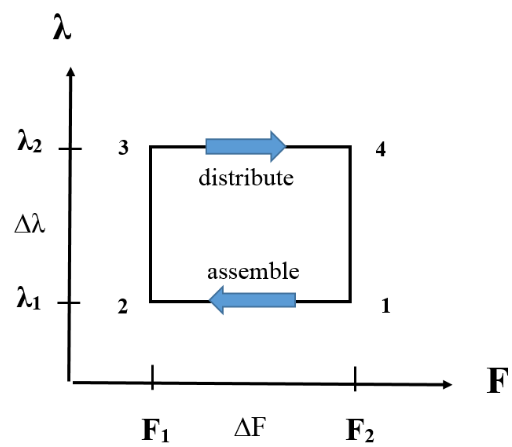


Fig. 6.1. The Carnot production circuit: Assemble products cheap and sell (distribute) expensive!

According to Carnot, we may divide the production process in fig. 6.1 into four sections:
1 → 2: Workers in a factory assemble or order (-d F) production parts at low labor costs (λ_1);

$$\delta L_1 = -\lambda_1 \, d F \qquad (6.3)$$

(L_1) is the value of labour input and depends on the amount of assembling (-d F). The standard of living of the workers (λ_1) stays constant during at least one cycle due to contracts. In order to produce at low costs many companies have production plants in countries with low standard of living, like China or India. Assembling at constant (λ) corresponds to an **isothermal** process in thermodynamics.

2 → 3: Products are refined in the company (at constant entropy F_1) to create a higher value (d λ) or are transported (exported) to a region, where the products have a higher value due to higher labor costs and standard of living (λ_2):

$$\delta L_2 = c N \, d \lambda \qquad (6.4)$$

During transport, export, import by car, train ship or plain products stay together and have constant entropy (F), this corresponds to an **adiabatic** process in thermodynamics.

3 → 4: at higher value the products are sold (distributed) to the customers,

$$\delta L_3 = \lambda_2 \, d F \qquad (6.5)$$

At selling or distributing the price (λ) stays constant, this is again called an **isothermal** process in thermodynamics.

4 → 1: after using the products for some time they will break, and they are transported to recycling centers. The value of the product has declined (d λ),

$$\delta L_4 = - c N \, d \lambda \qquad (6.6)$$

Transport e.g. in a container means constant entropy (F) and corresponds again to an **adiabatic** process in thermodynamics. At this point the production cycle starts again.

Monetary circuit of production δ M (λ, F)
According to double entry accounting the monetary and the productive circuit are both parts of the total balance,

$$\oint \delta M = \oint \lambda \, d F = (\lambda_2 - \lambda_1)\, \Delta F = Y_F - C_F \qquad (6.7).$$

Fig. 6.2 gives the monetary balance (δM) of production:

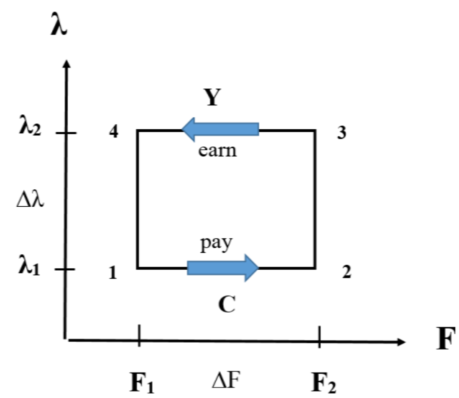


Fig. 6.2. The monetary circuit (δM) of Carnot production: Pay little for assembling and earn much by selling the products!

According to Carnot, the production process is again divided into four sections:

1 → 2: The factory pays workers little (λ_1) for assembling (-d F), the costs (C) are low;
2 → 3: products are refined to create a higher value (d λ) or are transported (exported) to a region, where the products are more valuable due to higher labor costs (λ_2);
3 → 4: The factory sells the products at higher value (λ_2) and earns the income (Y).
4 → 1: after selling the company starts again to produce more products.

According to eq. (6.1) monetary and productive circuits are equivalent.

7. Economic growth

The distribution of surplus

In the calculation of economic growth we cannot look directly for income Y (t) as a function of time (t), because (δ Y) does not have a stem function. Economic growth of households may be derived from the fundamental law of economics after n cycles:

$$-\oint \delta P = \oint \delta M = \mathrm{n} \oint \lambda \,\mathrm{d}\, \mathrm{F} = \mathrm{n}\, \Delta\, \mathrm{M} \qquad (7.1)$$

Production input (δP) generates surplus (ΔM_H) in each cycle, e. g. households go to work every day, and they earn 100 € and spend 90 € per day, generating a surplus of 10 € per day:

$$Y_H = \lambda_2 \Delta F = 100\text{ € / d} \qquad (7.2)$$

$$\underline{C_H = \lambda_1 \Delta F = 90\text{ € / d}} \qquad (7.3)$$

$$Y_H - C_H = \Delta M_H = (\lambda_2 - \lambda_1) \Delta F = 10\text{ € / d} \qquad (7.4)$$

The values of (λ_2) and (λ_1) are constant during each cycle. Households now have to decide what to do with the surplus in the next cycle: they may reinvest the surplus or consume it. Alternatively, they may split the surplus (ΔM_H) by raising income and consumption level after each circuit by

$$\Delta Y_H = \Delta \lambda_2 \Delta F = (1 - a) \Delta M_H \qquad (7.5)$$

$$\underline{\Delta C_H = \Delta \lambda_1 \Delta F = a \Delta M_H} \qquad (7.6)$$

$$\Delta Y_H + \Delta C_H = (1 - a + a) \Delta M_H = \Delta M_H \qquad (7.7)$$

The factor (a) is the percentage of surplus that is consumed, and (1 – a) is the rate of savings, which will be regarded as a constant in the following calculations.

Economic cycles and time

So far the economic laws in eq. (3.3) to (3.5) do not depend on time. Instead of counting the number (n) of cycles we may also introduce time (t),

$$n = \omega t \qquad (7.8).$$

The factor omega (ω) is the number of cycles per time interval. Workers may be paid once a day or once a week. Employees usually are paid once a month and farmers generally harvest once or twice a year. Industry goes by quarters of a year and countries by fiscal years.
The surplus (Δ M) of production and trade grows with the number (n) of cycles, the number of turns in the spring. In continuous production and trade surplus grows proportional to time (t), like the spiral in fig. 7.1:

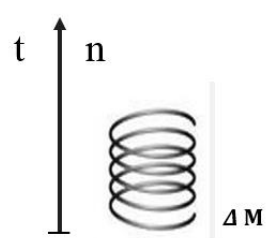


Fig. 7.1. Surplus (Δ M) grows with the number (n) of cycles of production and trade. In a continuous process surplus grows with frequency (ω) of the cycles and with time (t).

Rate of Savings
The rate of savings (1 – a) may be determined, if we plot the change of consumption (ΔC_H) as a function of rising income (ΔY_H) with time.

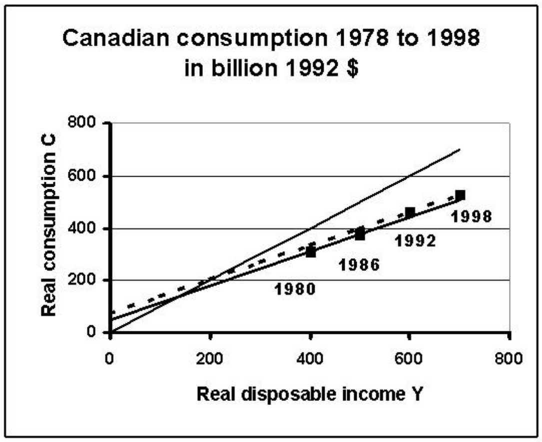


Fig. 7.2. Consumption function for Canada between 1978 and 1992. Consumption is a linear function of income (Y), the slope is m = 0,67, this corresponds to a = 0,40. The dashed line shows the rise of minimal costs (C_0) in Canada between 1980 and 1998.

Fig. 7.2 shows data for Canadian consumption (C_H) from 1978 to 1998 as a function of annual income of households (ΔY_H). Referring to eqs.(7.5) and (7.6) we obtain

$$\Delta C_H = [a / (1 – a)] \Delta Y_H \quad (7.9)$$

$$C_H = C_{H0} + [a / (1 – a)] \Delta Y_H \quad (7.10)$$

In fig. 7.1 the slope is [a / (1 – a)] = m = 0,67 = 2/3, this corresponds to a = 0,40 = 2/5.

Between 1978 and 1998 Canadians have consumed 40 % of their surplus,

$$\Delta C_H = a \Delta M_H = 0,4 \Delta M_H \quad (7.11)$$

$$\Delta Y_H = (1 – a) \Delta M_H = 0,6 \Delta M_H \quad (7.12)$$

and they have saved 60 % of their annual surplus.

The differential equations of growth
The distribution of surplus in eqs. (7.5) and (7.6) after n cycles may be written as a function of time,

$$\Delta Y_H = \Delta \lambda_2 \Delta F = n (1 – a) \Delta M_H = n (1 – a) (\lambda_2 – \lambda_1) \Delta F \quad (7.13)$$

$$\Delta C_H = \Delta \lambda_1 \Delta F = n a \Delta M_H = n a (\lambda_2 – \lambda_1) \Delta F \quad (7.14)$$

With d n = d ω t this leads to the following set of coupled differential equations for λ_1 and λ_2:

$$d \lambda_2 (t) = (1- a) (\lambda_2 - \lambda_1) d \omega t \quad (7.15).$$

$$d\lambda_1(t) = a\ (\lambda_2 - \lambda_1)\, d\omega t \qquad (7.16)$$

The solution of these equations is

$$\lambda_2(t) = \lambda_{20} + (1\text{-}a)\, [\lambda_{20} - \lambda_{10}]\, [\exp[(1\text{-}2a)\, \omega t] - 1] / (1\text{-}2a) \qquad (7.17).$$

$$\lambda_1(t) = \lambda_{10} + a\, [\lambda_{20} - \lambda_{10}]\, [\exp[(1\text{-}2a)\, \omega t] - 1] / (1\text{-}2a) \qquad (7.18).$$

Application to international trade

The results, eqs. (7.17) and (7.18) are presented for the distribution factor $a \leq 0,5$ in fig 7.1: (the distribution factor a is identical to the factor p in fig. 7.1. The factor a now determines the part of surplus that is distributed to party (1) and (1 – a) given to party (2)).

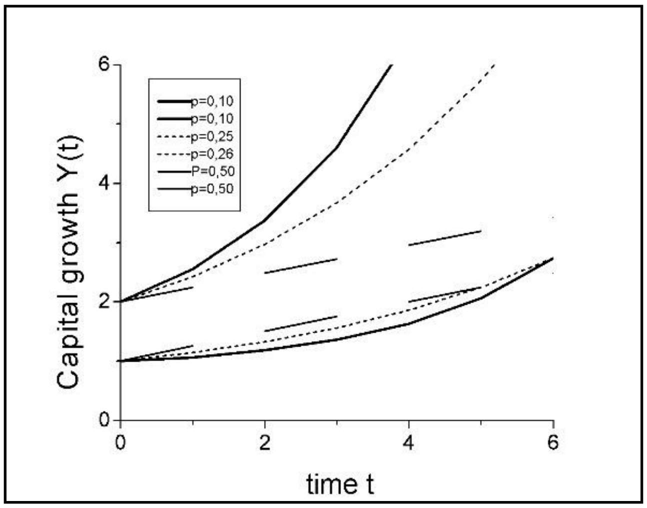


Fig. 7.3. The development of income of two interdependent economic systems starting at the levels $Y_{10} = 1$ and $Y_{20} = 2$ for three factors of distribution, a = 0,10; a = 0,25 and a = 0,50. For a = 0,10 both parties grow exponentially. For equal distribution of profits a = 0,5 both parties show linear growth. Surprisingly, the income of workers (Y_1) with a = 0,10 grows faster after some time than for workers with a = 0,25. This is due to the coupling to the fast rising industry (Y_2). At a = 0,10 industry rises fast with (1-a) = 0,90. At a larger participation a = 0,25 of the workers industry rises slower, and income of workers will rise slower, too.

a = 0,10: At 10 % of the profit for the poorer party (Y_1) and 90 % for the rich party (Y_2) both parties grow exponentially, fig. 7.1. Examples are Japan and Germany after World War II, both economies were depending on trade with USA and were growing exponentially. Another example today is the US – China trade relation, fig. 7.4.

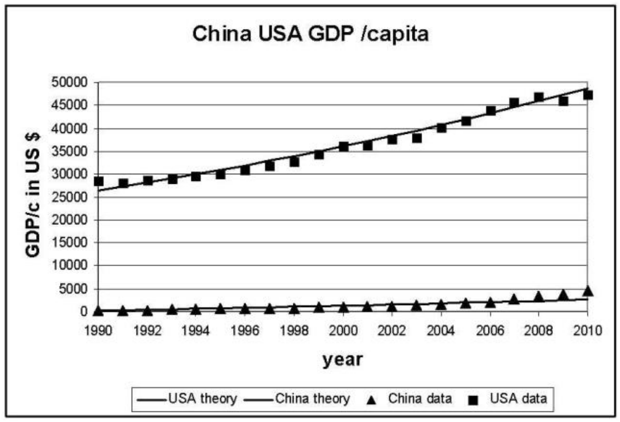


Fig. 7.4. Economic growth of a system of two parties (China and USA in 2006). The difference in GDP per capita, (Y_1) and (Y_2) is the basis of the economic motor between China and USA. Data from Index mundi 2012, calculation for a = 0,1 and ω = 0,035.

The difference in GDP per capita in China and USA is still very large and ensures big profits in trade between these two countries. The data between 1990 and 2010 have been fitted according to eqs. (7.3) and (7.4) and demonstrate that the model may very well be applied.

a = 0,25: At a more social participation in growth of 25 % for workers and 75 % for capital both incomes still grow exponentially. But as workers are linked to a less growing capital (Y_2), the income of workers (Y_1) with 25 % participation will growing less in longer times than for workers with 10 % participation, which are tied to a strong capital with 90 % participation, fig.7.1. This indicates clearly, that a high rise in wages may weaken the economy and lead to lower wages in the long run. A typical example may be France with strong unions and effective strike management.

a = 0,50: A surprising result is observed in fig. 21.1 for the even split of profits. An even split between the two parties seems to be a fair deal. A fair deal leads to linear growth for both parties. In companies with strong competition linear growth is not of interest for both parties. Accordingly, it seems generally not a good choice in long time wage negotiations. Nevertheless, as wage negotiations are only valid for short times, a fast rise of income seems always attractive for the poorer side. The same is true for managers with short time contracts. A short time manager will generally take the option of high short time linear growth with high personal benefits rather than a long time perspective of exponential growth of the company.

a > 0,5; For the distribution factor a > ½ the results are presented in fig 7.5:
a = 0,75: A factor a > 0,5 leads to decreasing growth of the system. In fig. 7.3. (Y_1) is trailing the decreasing (Y_2). After Japan has acquired many production plants, the factor p has grown and the efficiency of the exports started to decrease.

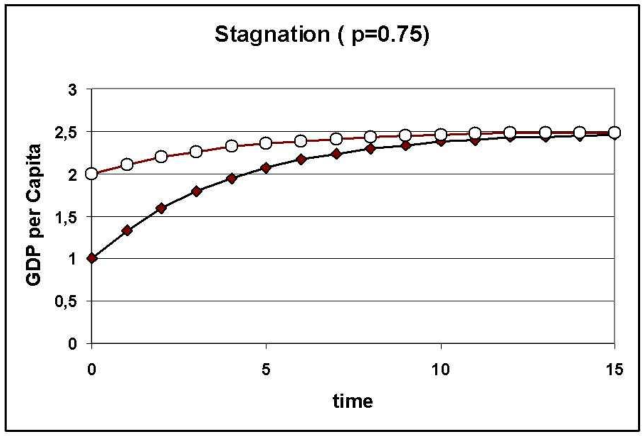


Fig. 7.5. The development of income of two interdependent economic systems starting at the levels $Y_{10} = 1$ and $Y_{20} = 2$ for the factors of distribution a > 0,50. This model leads to stagnation of income of the two interdependent economic systems stating at $Y_{10} = 1$ and Y_{20}. = 2. At high values of profit for the poor side, a = 0.75, economic growth is reaching a boundary with time for both partners.

The trade USA – Japan is presented in fig 7.5 for a = 0.51. The economic level of USA (Y_1) and Japan (Y_2) are trailing each other to a nearly constant final level. Japanese trade is in stagnation! US data agree with the model. Japanese data deviate from linear theory due to the strong variation of the Japanese currencies in this time. At high wages a country like Japan cannot export cheap labour anymore. Without resources, Japan can only sell innovative high tech products like e-cars to get out of stagnation.

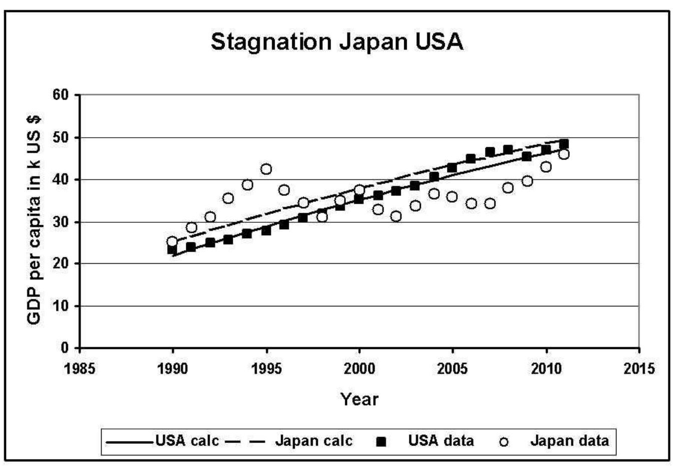


Fig. 7.6 The development of mean income (GDP / person in US $) of the interdependent relationship USA and Japan in quarters between 1990 and 2006. The US – Japan trade is at stagnation! US data agree with the model. Japanese data deviate from theory due to the strong variation of the Japanese currencies in this time. The efficiency of trade between the two countries is declining with time [World Bank, International Comparison Program database, 2012]. Calculations for a = 0.51 and ω = 0.9. International Monetary Fund's World Economic Outlook (WEO) Database, April 2012 Edition.

The growing gap between rich and poor

The higher the difference between income and costs, the higher the difference of income between capital and labour, the higher is the efficiency of trade and production. For a < ½ capital (λ_2) receives more than labour (λ_1) and the efficiency keeps growing. In a well working company and an industrialized society with p < ½ the difference between (λ_1) and (λ_2) keeps growing. This may observed in fig. 7.4: The difference of GDP / capita in China (λ_1) and the US (λ_2) grows with time. In relative number of the Chinese GDP / capita has multiplied by a factor of 4 from 1000 US $ / C in 1990 to 4000 US $ / C in 2010. The relative US GDP / capita has only multiplied by a factor of 1,7 from 28.000 US $ in 1990 to 47.000 US $ in 2010. But the difference in GDP per capita between China and USA has grown from 27.000 US $ in 1990 to 43.000 US $. The gap between rich and poor grows always for a < ½.

The same may be observed for all economic systems: In industrial societies with a < ½ the gap between rich and poor keeps growing, because a < ½ means more profit for the rich than for the poor. The same has been observed by Piketty [18]: The growth rate of capital has always been between 4 and 6 %, the growth rate of labour has always been between 0 and 4 %, as shown by Piketty [XX] in in fig. 7.7:

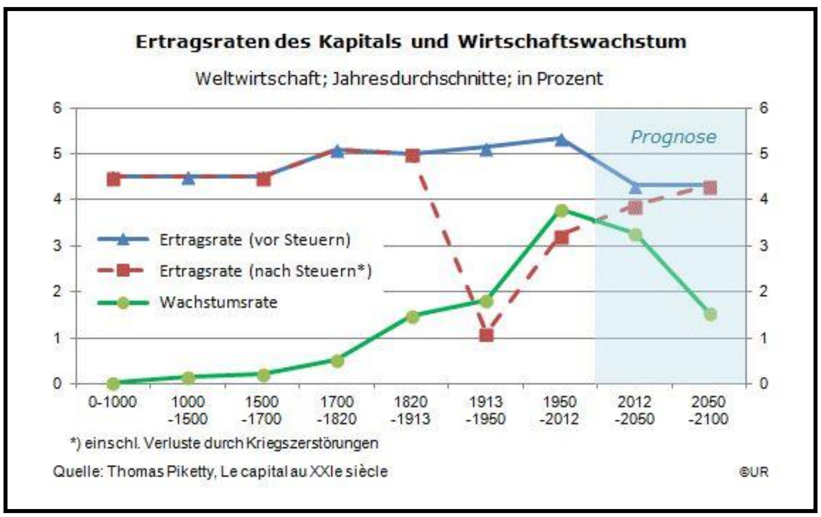


Fig. 7.7. Growth rate of capital (triangles, squares) and labour (circles) between the years 1000 and 2000. The growth rate of capital has always been between 4 and 6 %, the growth rate of labour has always been between 0 and 4 %. After Piketty [18].

The growing gap, the permanent advantage of the rich compared to the poorer side may seem unjustified. But what happens, if the poor side would get the permanent advantage, a > ½ ? This is observed in fig. 7.5. A permanent advantage of the poorer side, a > ½ means a growing income of labour and a declining income of capital. This leads to stagnation! The difference between (λ_1) and (λ_2), the economic efficiency, will go to zero! If we do this to a motor, the motor runs hot and will stop!

8. Conclusion

The future modl of science has guided to to calculus based econophysics with reasonable results in general economic theory as well as in applications to real data. This approach may be extended to micro-economics, finance, and to applications in social sciences and brings economics and social sciences closer to natural sciences and engineering [19 – 21].

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